3.4 Sum and Difference Identities

1. Answers:

a.

$$\cos\frac{5\pi}{12}=\cos\left(\frac{2\pi}{12}+\frac{3\pi}{12}\right)=\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\sin\frac{\pi}{4}$$
$$=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

b.

$$\cos\frac{7\pi}{12}=\cos\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right)=\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}=\frac{\sqrt{2}-\sqrt{6}}{4}$$

c.

$$\sin 345^\circ=\sin(300^\circ+45^\circ)=\sin 300^\circ\cos 45^\circ+\cos 300^\circ\sin 45^\circ$$
$$=-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=-\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}=\frac{\sqrt{2}-\sqrt{6}}{4}$$

d.

$$\tan 75^\circ=\tan(45^\circ+30^\circ)=\frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$$
$$=\frac{1+\frac{\sqrt{3}}{3}}{1-1\cdot\frac{\sqrt{3}}{3}}=\frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}}=\frac{3+\sqrt{3}}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}}=\frac{9+6\sqrt{3}+3}{9-3}=\frac{12+6\sqrt{3}}{6}=2+\sqrt{3}$$

e.

$$\cos 345^\circ=\cos(315^\circ+30^\circ)=\cos 315^\circ\cos 30^\circ-\sin 315^\circ\sin 30^\circ$$
$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-(-\frac{\sqrt{2}}{2})\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

f.

$$\sin\frac{17\pi}{12}=\sin\left(\frac{9\pi}{12}+\frac{8\pi}{12}\right)=\sin\left(\frac{3\pi}{4}+\frac{2\pi}{3}\right)=\sin\frac{3\pi}{4}\cos\frac{2\pi}{3}+\cos\frac{3\pi}{4}\sin\frac{2\pi}{3}$$
$$=\frac{\sqrt{2}}{2}\cdot(-\frac{1}{2})+-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}=-\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}=\frac{-\sqrt{2}-\sqrt{6}}{4}$$

2. If $\sin y=\frac{12}{13}$ and in Quadrant II, then by the Pythagorean Theorem $\cos y=-\frac{5}{13}(12^2+b^2=13^2)$.
   - And, if $\sin z=\frac{3}{5}$ and in Quadrant I, then by the Pythagorean Theorem $\cos z=\frac{4}{5}(a^2+3^2=5^2)$.
   - $\cos(y-z)=\cos y\cos z+\sin y\sin z$ and $=-\frac{5}{13}\cdot\frac{4}{5}+\frac{12}{13}\cdot\frac{3}{5}=-\frac{20}{65}+\frac{36}{65}=\frac{16}{65}$
3. If $\sin y=-\frac{5}{13}$ and in Quadrant III, then cosine is also negative.
   - By the Pythagorean Theorem, the second leg is $12(5^2+b^2=13^2)$, so $\cos y=-\frac{12}{13}$.
   - If the $\sin z=\frac{4}{5}$ and in Quadrant II, then the cosine is also negative.

- By the Pythagorean Theorem, the second leg is $3(4^2+b^2=5^2)$, so $\cos = -\frac{3}{5}$.
- To find $\sin(y+z)$, plug this information into the sine sum formula.

$$\begin{aligned}\sin(y+z) &= \sin y \cos z + \cos y \sin z\\ &= -\frac{5}{13}\cdot -\frac{3}{5} + -\frac{12}{13}\cdot\frac{4}{5} = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}\end{aligned}$$

4. Answers:
   a. This is the cosine difference formula, so: $\cos 80^\circ \cos 20^\circ + \sin 80^\circ 20^\circ = \cos(80^\circ - 20^\circ) = \cos 60^\circ = \frac{1}{2}$
   b. This is the expanded sine sum formula, so: $\sin 25^\circ \cos 5^\circ + \cos 25^\circ \sin 5^\circ = \sin(25^\circ + 5^\circ) = \sin 30^\circ = \frac{1}{2}$
5. Step 1: Expand using the cosine sum formula and change everything into sine and cosine

$$\begin{aligned}\frac{\cos(m-n)}{\sin m \cos n} &= \cot m + \tan n\\ \frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n} &= \frac{\cos m}{\sin m} + \frac{\sin n}{\cos n}\end{aligned}$$

Step 2: Find a common denominator for the right hand side.

$$= \frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n}$$

The two sides are the same, thus they are equal to each other and the identity is true.
6. $\cos(\pi+\theta) = \cos\pi\cos\theta - \sin\pi\sin\theta = -\cos\theta$
7. Step 1: Expand $\sin(a+b)$ and $\sin(a-b)$ using the sine sum and difference formulas. $\sin(a+b)\sin(a-b) = \cos^2 b - \cos^2 a$ $(\sin a \cos b + \cos a \sin b)(\sin a \cos b - \cos a \sin b)$ Step 2: FOIL and simplify.

$$\sin^2 a \cos^2 b - \sin a \cos a \sin b \cos b + \sin a \sin b \cos a \cos b - \cos^2 a \sin^2 b \sin^2 a \cos^2 b - \cos a^2 \sin^2 b$$

Step 3: Substitute $(1-\cos^2 a)$ for $\sin^2 a$ and $(1-\cos^2 b)$ for $\sin^2 b$, distribute and simplify.

$$\begin{aligned}&(1-\cos^2 a)\cos^2 b - \cos a^2(1-\cos^2 b)\\ &\cos^2 b - \cos^2 a \cos^2 b - \cos^2 a + \cos^2 a \cos^2 b\\ &\cos^2 b - \cos^2 a\end{aligned}$$

8. $\tan(\pi+\theta) = \frac{\tan\pi+\tan\theta}{1-\tan\pi\tan\theta} = \frac{\tan\theta}{1} = \tan\theta$
9. $\sin\frac{\pi}{2} = \sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{4} - \cos\frac{\pi}{4}\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{2}{4} - \frac{2}{4} = 0$ This could also be verified by using $60^\circ + 30^\circ$
10. Step 1: Expand using the cosine and sine sum formulas.

$$\cos(x+y)\cos y + \sin(x+y)\sin y = (\cos x \cos y - \sin x \sin y)\cos y + (\sin x \cos y + \cos x \sin y)\sin y$$

Step 2: Distribute $\cos y$ and $\sin y$ and simplify.

$$\begin{aligned}&= \cos x \cos^2 y - \sin x \sin y \cos y + \sin x \sin y \cos y + \cos x \sin^2 y\\ &= \cos x \cos^2 y + \cos x \sin^2 y\\ &= \cos x \underbrace{(\cos^2 y + \sin^2 y)}_{1}\\ &= \cos x\end{aligned}$$

11. Step 1: Expand left hand side using the sum and difference formulas

$$\frac{\cos(c+d)}{\cos(c-d)}=\frac{1-\tan c\tan d}{1+\tan c\tan d}$$
$$\frac{\cos c\cos d-\sin c\sin d}{\cos c\cos d+\sin c\sin d}=\frac{1-\tan c\tan d}{1+\tan c\tan d}$$

Step 2: Divide each term on the left side by $\cos c\cos d$ and simplify

$$\frac{\frac{\cos c\cos d}{\cos c\cos d}-\frac{\sin c\sin d}{\cos c\cos d}}{\frac{\cos c\cos d}{\cos c\cos d}+\frac{\sin c\sin d}{\cos c\cos d}}=\frac{1-\tan c\tan d}{1+\tan c\tan d}$$
$$\frac{1-\tan c\tan d}{1+\tan c\tan d}=\frac{1-\tan c\tan d}{1+\tan c\tan d}$$

12. To find all the solutions, between $[0,2\pi)$, we need to expand using the sum formula and isolate the $\cos x$.

$$2\cos^2\left(x+\frac{\pi}{2}\right)=1$$
$$\cos^2\left(x+\frac{\pi}{2}\right)=\frac{1}{2}$$
$$\cos\left(x+\frac{\pi}{2}\right)=\pm\sqrt{\frac{1}{2}}=\pm\frac{\sqrt{2}}{2}$$
$$\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2}=\pm\frac{\sqrt{2}}{2}$$
$$\cos x\cdot 0-\sin x\cdot 1=\pm\frac{\sqrt{2}}{2}$$
$$-\sin x=\pm\frac{\sqrt{2}}{2}$$
$$\sin x=\pm\frac{\sqrt{2}}{2}$$

This is true when $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4}$, or $\frac{7\pi}{4}$

13. First, solve for $\tan(x+\frac{\pi}{6})$.

$$2\tan^2\left(x+\frac{\pi}{6}\right)+1=7$$
$$2\tan^2\left(x+\frac{\pi}{6}\right)=6$$
$$\tan^2\left(x+\frac{\pi}{6}\right)=3$$
$$\tan\left(x+\frac{\pi}{6}\right)=\pm\sqrt{3}$$

Now, use the tangent sum formula to expand for when $\tan(x+\frac{\pi}{6})=\sqrt{3}$.

$$\frac{\tan x+\tan\frac{\pi}{6}}{1-\tan x\tan\frac{\pi}{6}}=\sqrt{3}$$
$$\tan x+\tan\frac{\pi}{6}=\sqrt{3}\left(1-\tan x\tan\frac{\pi}{6}\right)$$
$$\tan x+\frac{\sqrt{3}}{3}=\sqrt{3}-\sqrt{3}\tan x\cdot\frac{\sqrt{3}}{3}$$
$$\tan x+\frac{\sqrt{3}}{3}=\sqrt{3}-\tan x$$
$$2\tan x=\frac{2\sqrt{3}}{3}$$
$$\tan x=\frac{\sqrt{3}}{3}$$

This is true when $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. If the tangent sum formula to expand for when $\tan(x+\frac{\pi}{6}) = -\sqrt{3}$, we get no solution as shown.

$$\begin{aligned}
\frac{\tan x + \tan\frac{\pi}{6}}{1-\tan x\tan\frac{\pi}{6}} &= -\sqrt{3} \\
\tan x + \tan\frac{\pi}{6} &= -\sqrt{3}\left(1-\tan x\tan\frac{\pi}{6}\right) \\
\tan x + \frac{\sqrt{3}}{3} &= -\sqrt{3}+\sqrt{3}\tan x\cdot\frac{\sqrt{3}}{3} \\
\tan x + \frac{\sqrt{3}}{3} &= -\sqrt{3}+\tan x \\
\frac{\sqrt{3}}{3} &= -\sqrt{3}
\end{aligned}$$

Therefore, the tangent sum formula cannot be used in this case. However, since we know that $\tan(x+\frac{\pi}{6}) = -\sqrt{3}$ when $x+\frac{\pi}{6} = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$, we can solve for x as follows.

$$\begin{aligned}
x+\frac{\pi}{6} &= \frac{5\pi}{6} \\
x &= \frac{4\pi}{6} \\
x &= \frac{2\pi}{3}
\end{aligned}$$

$$\begin{aligned}
x+\frac{\pi}{6} &= \frac{11\pi}{6} \\
x &= \frac{10\pi}{6} \\
x &= \frac{5\pi}{3}
\end{aligned}$$

Therefore, all of the solutions are $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

14. To solve, expand each side:

$$\begin{aligned}
\sin\left(x+\frac{\pi}{6}\right) &= \sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x \\
\sin\left(x-\frac{\pi}{4}\right) &= \sin x\cos\frac{\pi}{4}-\cos x\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x
\end{aligned}$$

Set the two sides equal to each other:

$$\begin{aligned}
\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x &= \frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x \\
\sqrt{3}\sin x+\cos x &= \sqrt{2}\sin x-\sqrt{2}\cos x \\
\sqrt{3}\sin x-\sqrt{2}\sin x &= -\cos x-\sqrt{2}\cos x \\
\sin x\left(\sqrt{3}-\sqrt{2}\right) &= \cos x\left(-1-\sqrt{2}\right) \\
\frac{\sin x}{\cos x} &= \frac{-1-\sqrt{2}}{\sqrt{3}-\sqrt{2}} \\
\tan x &= \frac{-1-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} \\
&= \frac{-\sqrt{3}-\sqrt{2}+\sqrt{6}-2}{3-2} \\
&= -2+\sqrt{6}-\sqrt{3}-\sqrt{2}
\end{aligned}$$

As a decimal, this is -2.69677, so $\tan^{-1}(-2.69677)=x, x=290.35^\circ$ and 110.35°.