3.5 Double Angle Identities

1. If $\sin x = \frac{4}{5}$ and in Quadrant II, then cosine and tangent are negative. Also, by the Pythagorean Theorem, the third side is $3(b = \sqrt{5^2 - 4^2})$. So, $\cos x = -\frac{3}{5}$ and $\tan x = -\frac{4}{3}$. Using this, we can find $\sin 2x, \cos 2x$, and $\tan 2x$.

$$\begin{aligned} \sin 2x &= 2\sin x \cos x \\ &= 2 \cdot \frac{4}{5} \cdot -\frac{3}{5} \\ &= -\frac{24}{25} \end{aligned}$$

$$\begin{aligned} \cos 2x &= 1 - \sin^2 x \\ &= 1 - 2 \cdot \left(\frac{4}{5}\right)^2 \\ &= 1 - 2 \cdot \frac{16}{25} \\ &= 1 - \frac{32}{25} \\ &= -\frac{7}{25} \end{aligned}$$

$$\begin{aligned} \tan 2x &= \frac{2\tan x}{1 - \tan^2 x} \\ &= \frac{2 \cdot -\frac{4}{3}}{1 - \left(-\frac{4}{3}\right)^2} \\ &= \frac{-\frac{8}{3}}{1 - \frac{16}{9}} = -\frac{8}{3} \div -\frac{7}{9} \\ &= -\frac{8}{3} \cdot -\frac{9}{7} \\ &= \frac{24}{7} \end{aligned}$$

2. This is one of the forms for $\cos 2x$.

$$\begin{aligned} \cos^2 15^\circ - \sin^2 15^\circ &= \cos(15^\circ \cdot 2) \\ &= \cos 30^\circ \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

3. Step 1: Use the cosine sum formula

$$\begin{aligned} \cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ \cos(2\theta + \theta) &= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \end{aligned}$$

Step 2: Use double angle formulas for $\cos 2\theta$ and $\sin 2\theta$

$$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$

Step 3: Distribute and simplify.

$$\begin{aligned} &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\ &= -\cos\theta(-2\cos^2\theta + 2\sin^2\theta + 1) \\ &= -\cos\theta[-2\cos^2\theta + 2(1 - \cos^2\theta) + 1] && \rightarrow \text{Substitute } 1 - \cos^2\theta \text{ for } \sin^2\theta \\ &= -\cos\theta[-2\cos^2\theta + 2 - 2\cos^2\theta + 1] \\ &= -\cos\theta(-4\cos^2\theta + 3) \\ &= 4\cos^3\theta - 3\cos\theta \end{aligned}$$

4. Step 1: Expand $\sin 2t$ using the double angle formula.

$$\begin{aligned} \sin 2t - \tan t &= \tan t \cos 2t \\ 2\sin t\cos t - \tan t &= \tan t \cos 2t \end{aligned}$$

Step 2: change $\tan t$ and find a common denominator.

$$\begin{gathered}
2\sin t\cos t-\frac{\sin t}{\cos t}\\
\frac{2\sin t\cos^2 t-\sin t}{\cos t}\\
\frac{\sin t(2\cos^2 t-1)}{\cos t}\\
\frac{\sin t}{\cos t}\cdot(2\cos^2 t-1)\\
\tan t\cos 2t
\end{gathered}$$

5. If $\sin x=-\frac{9}{41}$ and in Quadrant III, then $\cos x=-\frac{40}{41}$ and $\tan x=\frac{9}{40}$ (Pythagorean Theorem, $b=\sqrt{41^2-(-9)^2}$). So,

$$\begin{aligned}
\sin 2x&=2\sin x\cos x\\
&=2\cdot-\frac{9}{41}\cdot-\frac{40}{41}\\
&=\frac{720}{1681}
\end{aligned}
\qquad
\begin{aligned}
\cos 2x&=2\cos^2 x-1\\
&=2\left(-\frac{40}{41}\right)^2-1\\
&=\frac{3200}{1681}-\frac{1681}{1681}\\
&=\frac{1519}{1681}
\end{aligned}
\qquad
\begin{aligned}
\tan 2x&=\frac{\sin 2x}{\cos 2x}\\
&=\frac{\frac{720}{1681}}{\frac{1519}{1681}}\\
&=\frac{720}{1519}
\end{aligned}$$

6. Step 1: Expand $\sin 2x$

$$\begin{aligned}
\sin 2x+\sin x&=0\\
2\sin x\cos x+\sin x&=0\\
\sin x(2\cos x+1)&=0
\end{aligned}$$

Step 2: Separate and solve each for x.

$$\begin{aligned}
\sin x&=0\\
x&=0,\pi
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
2\cos x+1&=0\\
\cos x&=-\frac{1}{2}\\
x&=\frac{2\pi}{3},\frac{4\pi}{3}
\end{aligned}$$

7. Expand $\cos 2x$ and simplify

$$\begin{aligned}
\cos^2 x-\cos 2x&=0\\
\cos^2 x-(2\cos^2 x-1)&=0\\
-\cos^2 x+1&=0\\
\cos^2 x&=1\\
\cos x&=\pm 1
\end{aligned}$$

$\cos x=1$ when $x=0$, and $\cos x=-1$ when $x=\pi$. Therefore, the solutions are $x=0,\pi$.

8. a. 3.429 b. 0.960 c. 0.280

9. a.

$$\begin{aligned}
2\csc x\ 2x &= \frac{2}{\sin 2x} \\
2\csc x\ 2x &= \frac{2}{2\sin x\cos x} \\
2\csc x\ 2x &= \frac{1}{\sin x\cos x} \\
2\csc x\ 2x &= \left(\frac{\sin x}{\sin x}\right)\left(\frac{1}{\sin x\cos x}\right) \\
2\csc x\ 2x &= \frac{\sin x}{\sin^2 x\cos x} \\
2\csc x\ 2x &= \frac{1}{\sin^2 x}\cdot\frac{\sin x}{\cos x} \\
2\csc x\ 2x &= \csc^2 x\tan x
\end{aligned}$$

b.

$$\begin{aligned}
\cos^4\theta-\sin^4\theta &= (\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta) \\
\cos^4-\sin^4\theta &= 1(\cos^2\theta-\sin^2\theta) \\
\cos 2\theta &= \cos^2\theta-\sin^2\theta \\
\therefore \cos^4\theta-\sin^4\theta &= \cos 2\theta
\end{aligned}$$

c.

$$\begin{aligned}
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x\cos x}{1+(1-2\sin^2 x)} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x\cos x}{2-2\sin^2 x} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x\cos x}{2(1-\sin^2 x)} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x\cos x}{2\cos^2 x} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{\sin x}{\cos x} \\
\frac{\sin 2x}{1+\cos 2x} &= \tan x
\end{aligned}$$

10. $\cos 2x-1=\sin^2 x$

$$\begin{aligned}
(1-2\sin^2 x)-1 &= \sin^2 x \\
-2\sin^2 x &= \sin^2 x \\
0 &= 3\sin^2 x \\
0 &= \sin^2 x \\
0 &= \sin x \\
x &= 0,\pi
\end{aligned}$$

11.

$$\begin{aligned}
\cos 2x &= \cos x \\
2\cos^2 x - 1 &= \cos x \\
2\cos^2 x - \cos x - 1 &= 0 \\
(2\cos x + 1)(\cos x - 1) &= 0 \\
&\searrow \qquad \searrow \\
2\cos x + 1 = 0 \quad &\text{or} \quad \cos x - 1 = 0 \\
2\cos x = -1 \qquad & \qquad \cos x = 1 \\
\cos x = -\frac{1}{2} &
\end{aligned}$$

$\cos x = 1$ when $x = 0$ and $\cos x = -\frac{1}{2}$ when $x = \frac{2\pi}{3}$.

12.

$$\begin{aligned}
2\csc 2x \tan x &= \sec^2 x \\
\frac{2}{\sin 2x} \cdot \frac{\sin x}{\cos x} &= \frac{1}{\cos^2 x} \\
\frac{2}{2\sin x \cos x} \cdot \frac{\sin x}{\cos x} &= \frac{1}{\cos^2 x} \\
\frac{1}{\cos^2 x} &= \frac{1}{\cos^2 x}
\end{aligned}$$

13. $\sin 2x - \cos 2x = 1$

$$\begin{aligned}
2\sin x \cos x - (1 - 2\sin^2 x) &= 1 \\
2\sin x \cos x - 1 + 2\sin^2 x &= 1 \\
2\sin x \cos x + 2\sin^2 x &= 2 \\
\sin x \cos x + \sin^2 x &= 1 \\
\sin x \cos x &= 1 - \sin^2 x \\
\sin x \cos x &= \cos^2 x \\
\left(\pm\sqrt{1-\cos^2 x}\right)\cos x &= \cos^2 x \\
\left(1 - \cos^2 x\right)\cos^2 x &= \cos^4 x \\
\cos^2 x - \cos^4 x &= \cos^4 x \\
\cos^2 x - 2\cos^4 x &= 0 \\
\cos^2 x(1 - 2\cos^2 x) &= 0 \\
\swarrow \qquad & \qquad \searrow \\
& 1 - 2\cos^2 x = 0 \\
\cos^2 x = 0 \qquad & -2\cos^2 x = -1 \\
\cos x = 0 \qquad \text{or} \qquad & \cos^2 x = \frac{1}{2} \\
x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad & \cos x = \pm\frac{\sqrt{2}}{2} \\
& x = \frac{\pi}{4}, \frac{5\pi}{4}
\end{aligned}$$

Note: If we go back to the equation $\sin x \cos x = \cos^2 x$, we can see that $\sin x \cos x$ must be positive or zero, since $\cos^2 x$ is always positive or zero. For this reason, $\sin x$ and $\cos x$ must have the same sign (or one of them must be zero), which means that x cannot be in the second or fourth quadrants. This is why $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ are not valid solutions.

14. Use the double angle identity for $\cos 2x$.

$$\begin{aligned}
\sin^2 x - 2 &= \cos 2x \\
\sin^2 x - 2 &= \cos 2x \\
\sin^2 x - 2 &= 1 - 2\sin^2 x \\
3\sin^2 x &= 3 \\
\sin^2 x &= 1 \\
\sin x &= \pm 1 \\
x &= \frac{\pi}{2}, \frac{3\pi}{2}
\end{aligned}$$