3.6 Half-Angle Identities

1. Answers:

a.

$$\cos 112.5^\circ = \cos \frac{225^\circ}{2} = -\sqrt{\frac{1+\cos 225^\circ}{2}}$$
$$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{4}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$$

b.

$$\sin 105^\circ = \sin \frac{210^\circ}{2} = \sqrt{\frac{1-\cos 210^\circ}{2}}$$
$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

c.

$$\tan \frac{7\pi}{8} = \tan \frac{1}{2} \cdot \frac{7\pi}{4} = \frac{1-\cos \frac{7\pi}{4}}{\sin \frac{7\pi}{4}}$$
$$= \frac{1-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\frac{2-\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -\frac{2-\sqrt{2}}{\sqrt{2}} = \frac{-2\sqrt{2}+2}{2} = -\sqrt{2}+1$$

d. $\tan \frac{\pi}{8} = \tan \frac{1}{2} \cdot \frac{\pi}{4} = \frac{1-\cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\frac{2-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-2}{2} = \sqrt{2}-1$

e. $\sin 67.5^\circ = \sin \frac{135^\circ}{2} = \sqrt{\frac{1-\cos 135^\circ}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

f. $\tan 165^\circ = \tan \frac{330^\circ}{2} = \frac{1-\cos 330^\circ}{\sin 330^\circ} = \frac{1-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\frac{2-\sqrt{3}}{2}}{-\frac{1}{2}} = -\left(2-\sqrt{3}\right) = -2+\sqrt{3}$

2. If $\sin\theta = \frac{7}{25}$, then by the Pythagorean Theorem the third side is 24. Because θ is in the second quadrant, $\cos\theta = -\frac{24}{25}$.

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1+\frac{24}{25}}{2}} = \sqrt{\frac{49}{50}} = \frac{7}{5\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{10}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1-\frac{24}{25}}{2}} = \sqrt{\frac{1}{50}} = \frac{1}{5\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{10}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1+\frac{24}{25}}{1-\frac{24}{25}}} = \sqrt{\frac{49}{50}\cdot\frac{50}{1}} = \sqrt{49} = 7$$

3. Step 1: Change right side into sine and cosine.

$$\begin{aligned}
\tan\frac{b}{2} &= \frac{\sec b}{\sec b\csc b+\csc b} \\
&= \frac{1}{\cos b}\div\csc b(\sec b+1) \\
&= \frac{1}{\cos b}\div\frac{1}{\sin b}\left(\frac{1}{\cos b}+1\right) \\
&= \frac{1}{\cos b}\div\frac{1}{\sin b}\left(\frac{1+\cos b}{\cos b}\right) \\
&= \frac{1}{\cos b}\div\frac{1+\cos b}{\sin b\cos b} \\
&= \frac{1}{\cos b}\cdot\frac{\sin b\cos b}{1+\cos b} \\
&= \frac{\sin b}{1+\cos b}
\end{aligned}$$

Step 2: At the last step above, we have simplified the right side as much as possible, now we simplify the left side, using the half angle formula.

$$\begin{aligned}
\sqrt{\frac{1-\cos b}{1+\cos b}} &= \frac{\sin b}{1+\cos b} \\
\frac{1-\cos b}{1+\cos b} &= \frac{\sin^2 b}{(1+\cos b)^2} \\
(1-\cos b)(1+\cos b)^2 &= \sin^2 b(1+\cos b) \\
(1-\cos b)(1+\cos b) &= \sin^2 b \\
1-\cos^2 b &= \sin^2 b
\end{aligned}$$

4. Step 1: change cotangent to cosine over sine, then cross-multiply.

$$\begin{aligned}
\cot\frac{c}{2} &= \frac{\sin c}{1-\cos c} \\
&= \frac{\cos\frac{c}{2}}{\sin\frac{c}{2}} = \sqrt{\frac{1+\cos c}{1-\cos c}} \\
\sqrt{\frac{1+\cos c}{1-\cos c}} &= \frac{\sin c}{1-\cos c} \\
\frac{1+\cos c}{1-\cos c} &= \frac{\sin^2 c}{(1-\cos c)^2} \\
(1+\cos c)(1-\cos c)^2 &= \sin^2 c(1-\cos c) \\
(1+\cos c)(1-\cos c) &= \sin^2 c \\
1-\cos^2 c &= \sin^2 c
\end{aligned}$$

5.

$$\begin{aligned}
\sin x\tan\frac{x}{2}+2\cos x &= \sin x\left(\frac{1-\cos x}{\sin x}\right)+2\cos x \\
\sin x\tan\frac{x}{2}+2\cos x &= 1-\cos x+2\cos x \\
\sin x\tan\frac{x}{2}+2\cos x &= 1+\cos x \\
\sin x\tan\frac{x}{2}+2\cos x &= 2\cos^2\frac{x}{2}
\end{aligned}$$

6. First, we need to find the third side.

- Using the Pythagorean Theorem, we find that the final side is $\sqrt{105}\left(b=\sqrt{13^2-(-8)^2}\right)$.
- Using this information, we find that $\cos u = \pm\frac{\sqrt{105}}{13}$.
- Plugging this into the half angle formula, we get:

$$\begin{aligned}\cos\frac{u}{2} &= -\sqrt{\frac{1\pm\sqrt{\frac{105}{13}}}{2}}\\ &= -\sqrt{\frac{\frac{13\pm\sqrt{105}}{13}}{2}}\\ &= -\sqrt{\frac{13\pm\sqrt{105}}{26}}\end{aligned}$$

7. To solve $2\cos^2\frac{x}{2}=1$, first we need to isolate cosine, then use the half angle formula.

$$\begin{aligned}2\cos^2\frac{x}{2} &= 1\\ \cos^2\frac{x}{2} &= \frac{1}{2}\\ \frac{1+\cos x}{2} &= \frac{1}{2}\\ 1+\cos x &= 1\\ \cos x &= 0\end{aligned}$$

$\cos x = 0$ when $x=\frac{\pi}{2},\frac{3\pi}{2}$

8. To solve $\tan\frac{a}{2}=4$, first isolate tangent, then use the half angle formula.

$$\begin{aligned}\tan\frac{a}{2} &= 4\\ \sqrt{\frac{1-\cos a}{1+\cos a}} &= 4\\ \frac{1-\cos a}{1+\cos a} &= 16\\ 16+16\cos a &= 1-\cos a\\ 17\cos a &= -15\\ \cos a &= -\frac{15}{17}\end{aligned}$$

Using your graphing calculator, $\cos a = -\frac{15}{17}$ when $a = 152^\circ, 208^\circ$

9.

$$\begin{aligned}
\cos\frac{x}{2} &= 1+\cos x \\
\pm\sqrt{\frac{1+\cos x}{2}} &= 1+\cos x && \text{Half angle identity} \\
\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2 &= (1+\cos x)^2 && \text{square both sides} \\
\frac{1+\cos x}{2} &= 1+2\cos x+\cos^2 x \\
2\left(\frac{1+\cos x}{2}\right) &= 2(1+2\cos x+\cos^2 x) \\
1+\cos x &= 2+4\cos x+2\cos^2 x \\
2\cos^2 x+3\cos x+1 &= 0 \\
(2\cos x+1)(\cos x+1) &= 0 \\
\text{Then}\quad 2\cos x+1 &= 0 \\
\frac{2\cos x}{2} &= \frac{-1}{2} \\
x &= \frac{2\pi}{3}, \frac{4\pi}{3} \\
\text{Or}\quad \cos x+1 &= 0 \\
\cos x &= -1 \\
x &= \pi
\end{aligned}$$

10. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$ This is the two formulas for $\tan\frac{x}{2}$. Cross-multiply.

$$\begin{aligned}
\frac{\sin x}{1+\cos x} &= \frac{1-\cos x}{\sin x} \\
(1-\cos x)(1+\cos x) &= \sin^2 x \\
1+\cos x-\cos x-\cos^2 x &= \sin^2 x \\
1-\cos^2 x &= \sin^2 x \\
1 &= \sin^2 x+\cos^2 x
\end{aligned}$$