

### Section 3.4 Rational Functions

In the last few sections, we have built polynomials based on the positive whole number power functions. In this section we explore functions based on power functions with negative integer powers, called rational functions.

#### Example 1

You plan to drive 100 miles. Find a formula for the time the trip will take as a function of the speed you drive.

You may recall that multiplying speed by time will give you distance. If we let  $t$  represent the drive time in hours, and  $v$  represent the velocity (speed or rate) at which we drive, then  $vt = \text{distance}$ . Since our distance is fixed at 100 miles,  $vt = 100$ .

Solving this relationship for the time gives us the function we desired:

$$t(v) = \frac{100}{v} = 100v^{-1}$$

While this type of relationship can be written using the negative exponent, it is more common to see it written as a fraction.

This particular example is one of an **inversely proportional** relationship – where one quantity is a constant divided by the other quantity, like  $y = \frac{5}{x}$ .

Notice that this is a transformation of the reciprocal toolkit function,  $f(x) = \frac{1}{x}$ .

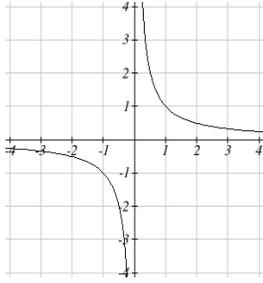
Several natural phenomena, such as gravitational force and volume of sound, behave in a manner **inversely proportional to the square** of another quantity. For example, the

volume,  $V$ , of a sound heard at a distance  $d$  from the source would be related by  $V = \frac{k}{d^2}$

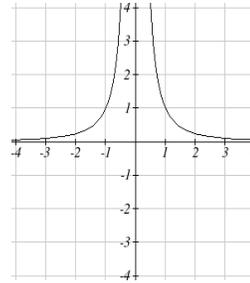
for some constant value  $k$ .

These functions are transformations of the reciprocal squared toolkit function  $f(x) = \frac{1}{x^2}$ .

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. These graphs have several important features.



$$f(x) = \frac{1}{x}$$



$$f(x) = \frac{1}{x^2}$$

Let's begin by looking at the reciprocal function,  $f(x) = \frac{1}{x}$ . As you well know, dividing by zero is not allowed and therefore zero is not in the domain, and so the function is undefined at an input of zero.

#### Short run behavior:

As the input values approach zero from the left side (taking on very small, negative values), the function values become very large in the negative direction (in other words, they approach negative infinity).

We write: as  $x \rightarrow 0^-$ ,  $f(x) \rightarrow -\infty$ .

As we approach zero from the right side (small, positive input values), the function values become very large in the positive direction (approaching infinity).

We write: as  $x \rightarrow 0^+$ ,  $f(x) \rightarrow \infty$ .

This behavior creates a **vertical asymptote**. An asymptote is a line that the graph approaches. In this case the graph is approaching the vertical line  $x = 0$  as the input becomes close to zero.

#### Long run behavior:

As the values of  $x$  approach infinity, the function values approach 0.

As the values of  $x$  approach negative infinity, the function values approach 0.

Symbolically: as  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow 0$

Based on this long run behavior and the graph we can see that the function approaches 0 but never actually reaches 0, it just "levels off" as the inputs become large. This behavior creates a **horizontal asymptote**. In this case the graph is approaching the horizontal line  $f(x) = 0$  as the input becomes very large in the negative and positive directions.

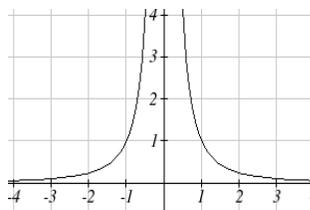
### Vertical and Horizontal Asymptotes

A **vertical asymptote** of a graph is a vertical line  $x = a$  where the graph tends towards positive or negative infinity as the inputs approach  $a$ . As  $x \rightarrow a$ ,  $f(x) \rightarrow \pm\infty$ .

A **horizontal asymptote** of a graph is a horizontal line  $y = b$  where the graph approaches the line as the inputs get large. As  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow b$ .

## Try it Now:

1. Use symbolic notation to describe the long run behavior and short run behavior for the reciprocal squared function.



## Example 2

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Transforming the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3, \text{ or equivalently, by giving the terms a common denominator,}$$

$$f(x) = \frac{3x+7}{x+2}$$

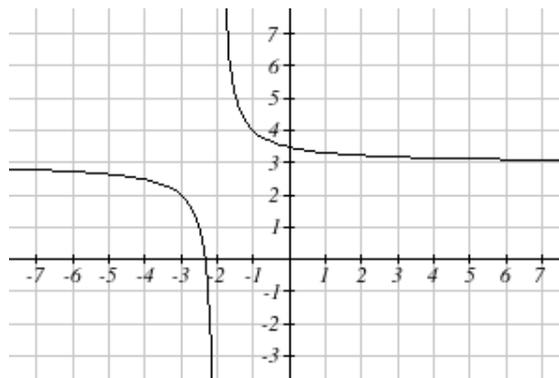
Shifting the toolkit function would give us this graph. Notice that this equation is undefined at  $x = -2$ , and the graph also is showing a vertical asymptote at  $x = -2$ .

As  $x \rightarrow -2^-$ ,  $f(x) \rightarrow -\infty$ , and as

$$x \rightarrow -2^+, f(x) \rightarrow \infty$$

As the inputs grow large, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at  $y = 3$ .

As  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow 3$ .



Notice that horizontal and vertical asymptotes get shifted left 2 and up 3 along with the function.

## Try it Now

2. Sketch the graph and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

In the previous example, we shifted a toolkit function in a way that resulted in a function of the form  $f(x) = \frac{3x+7}{x+2}$ . This is an example of a more general rational function.

### Rational Function

A **rational function** is a function that can be written as the ratio of two polynomials,  $P(x)$  and  $Q(x)$ .

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_px^p}{b_0 + b_1x + b_2x^2 + \cdots + b_qx^q}$$

### Example 3

A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after  $t$  minutes.

Notice that the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\text{water} = 100 + 10t$$

$$\text{sugar} = 5 + 1t$$

The concentration,  $C$ , will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5 + t}{100 + 10t}$$

### Finding Asymptotes and Intercepts

Given a rational function, as part of investigating the short run behavior we are interested in finding any vertical and horizontal asymptotes, as well as finding any vertical or horizontal intercepts, as we have done in the past.

To find vertical asymptotes, we notice that the vertical asymptotes in our examples occur when the denominator of the function is undefined. With one exception, a vertical asymptote will occur whenever the denominator is undefined.

### Example 4

Find the vertical asymptotes of the function  $k(x) = \frac{5 + 2x^2}{2 - x - x^2}$

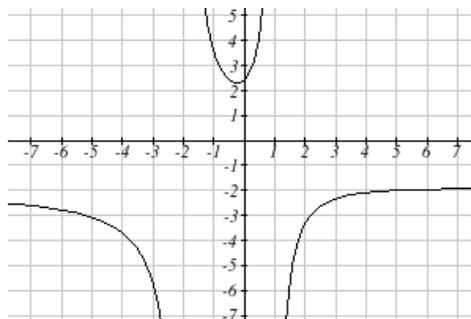
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$2 - x - x^2 = 0$$

$$(2 + x)(1 - x) = 0$$

$$x = -2, 1$$

This indicates two vertical asymptotes, which a look at a graph confirms.



The exception to this rule can occur when both the numerator and denominator of a rational function are zero at the same input.

#### Example 5

Find the vertical asymptotes of the function  $k(x) = \frac{x-2}{x^2-4}$ .

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

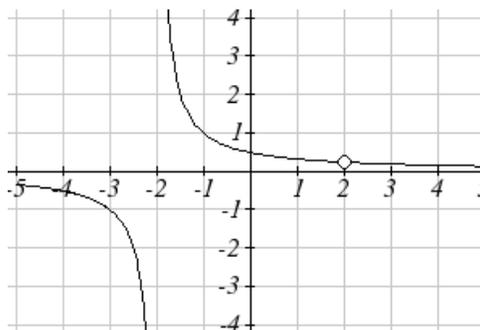
$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = -2, 2$$

However, the numerator of this function is also equal to zero when  $x = 2$ . Because of this, the function will still be undefined at 2, since  $\frac{0}{0}$  is undefined, but the graph will not have a vertical asymptote at  $x = 2$ .

The graph of this function will have the vertical asymptote at  $x = -2$ , but at  $x = 2$  the graph will have a hole: a single point where the graph is not defined, indicated by an open circle.



#### Vertical Asymptotes and Holes of Rational Functions

The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.

A **hole** might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. In this case, factor the numerator and denominator and simplify; if the simplified expression still has a zero in the denominator at the original input the original function has a vertical asymptote at the input, otherwise it has a hole.

To find horizontal asymptotes, we are interested in the behavior of the function as the input grows large, so we consider long run behavior of the numerator and denominator separately. Recall that a polynomial's long run behavior will mirror that of the leading term. Likewise, a rational function's long run behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

There are three distinct outcomes when this analysis is done:

**Case 1:** The degree of the denominator  $>$  degree of the numerator

Example:  $f(x) = \frac{3x + 2}{x^2 + 4x - 5}$

In this case, the long run behavior is  $f(x) \approx \frac{3x}{x^2} = \frac{3}{x}$ . This tells us that as the inputs grow

large, this function will behave similarly to the function  $g(x) = \frac{3}{x}$ . As the inputs grow

large, the outputs will approach zero, resulting in a horizontal asymptote at  $y = 0$ .

As  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow 0$

**Case 2:** The degree of the denominator  $<$  degree of the numerator

Example:  $f(x) = \frac{3x^2 + 2}{x - 5}$

In this case, the long run behavior is  $f(x) \approx \frac{3x^2}{x} = 3x$ . This tells us that as the inputs

grow large, this function will behave similarly to the function  $g(x) = 3x$ . As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote.

As  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow \pm\infty$ , respectively.

Ultimately, if the numerator is larger than the denominator, the long run behavior of the graph will mimic the behavior of the reduced long run behavior fraction. As another

example if we had the function  $f(x) = \frac{3x^5 - x^2}{x + 3}$  with long run behavior

$f(x) \approx \frac{3x^5}{x} = 3x^4$ , the long run behavior of the graph would look similar to that of an

even polynomial, and as  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow \infty$ .

**Case 3:** The degree of the denominator  $=$  degree of the numerator

Example:  $f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$

In this case, the long run behavior is  $f(x) \approx \frac{3x^2}{x^2} = 3$ . This tells us that as the inputs grow large, this function will behave like the function  $g(x) = 3$ , which is a horizontal line. As  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow 3$ , resulting in a horizontal asymptote at  $y = 3$ .

### Horizontal Asymptote of Rational Functions

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

Degree of denominator > degree of numerator: Horizontal asymptote at  $y = 0$

Degree of denominator < degree of numerator: No horizontal asymptote

Degree of denominator = degree of numerator: Horizontal asymptote at ratio of leading coefficients.

### Example 6

In the sugar concentration problem from earlier, we created the equation

$$C(t) = \frac{5 + t}{100 + 10t}.$$

Find the horizontal asymptote and interpret it in context of the scenario.

Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is  $t$ , with coefficient 1. In the denominator, the leading term is  $10t$ , with coefficient 10. The horizontal asymptote will be at the ratio of these

values: As  $t \rightarrow \infty$ ,  $C(t) \rightarrow \frac{1}{10}$ . This function will have a horizontal asymptote at

$$y = \frac{1}{10}.$$

This tells us that as the input gets large, the output values will approach  $1/10$ . In context, this means that as more time goes by, the concentration of sugar in the tank will approach one tenth of a pound of sugar per gallon of water or  $1/10$  pounds per gallon.

### Example 7

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

First, note this function has no inputs that make both the numerator and denominator zero, so there are no potential holes. The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at  $x = 1, -2, \text{ and } 5$ , indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow 0$ . This function will have a horizontal asymptote at  $y = 0$ .

### Try it Now

3. Find the vertical and horizontal asymptotes of the function

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

### Intercepts

As with all functions, a rational function will have a vertical intercept when the input is zero, if the function is defined at zero. It is possible for a rational function to not have a vertical intercept if the function is undefined at zero.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be zero (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. Since a fraction is only equal to zero when the numerator is zero, horizontal intercepts will occur when the numerator of the rational function is equal to zero.

### Example 8

Find the intercepts of  $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

We can find the vertical intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = -\frac{3}{5}$$

The horizontal intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \quad \text{This is zero when the numerator is zero}$$

$$0 = (x-2)(x+3)$$

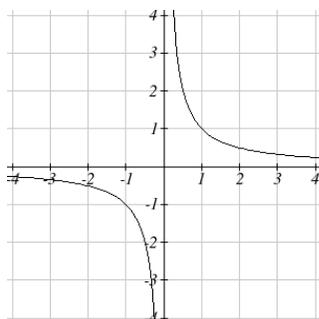
$$x = 2, -3$$

### Try it Now

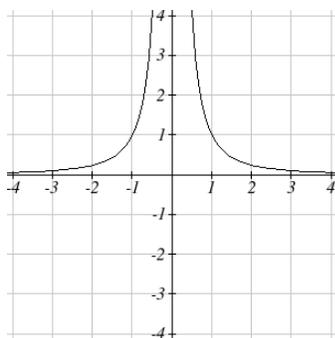
4. Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function and find the horizontal and vertical intercepts and the horizontal and vertical asymptotes.

From the previous example, you probably noticed that the numerator of a rational function reveals the horizontal intercepts of the graph, while the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Happily, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

When factors of the denominator have integer powers greater than one, the behavior at the corresponding vertical asymptote will mirror one of the two toolkit reciprocal functions.



We get this behavior when the degree of the factor in the denominator is odd. The distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity.



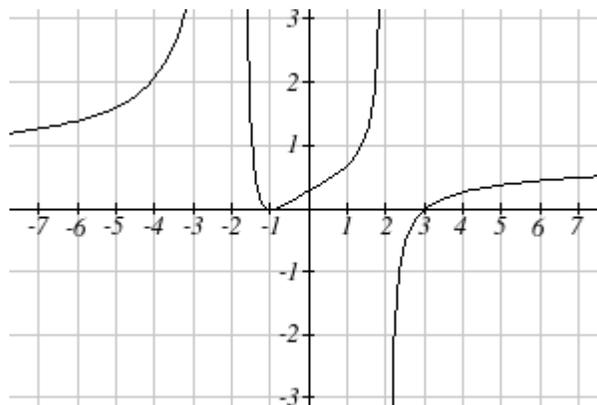
We get this behavior when the degree of the factor in the denominator is even. The distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote, or heads toward negative infinity on both sides.

For example, the graph of

$$f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$$

is shown here.

At the horizontal intercept  $x = -1$  corresponding to the  $(x+1)^2$  factor of the numerator, the graph bounces at the intercept, consistent with the quadratic nature of the factor.



At the horizontal intercept  $x = 3$  corresponding to the  $(x-3)$  factor of the numerator, the graph passes through the axis as we'd expect from a linear factor.

At the vertical asymptote  $x = -3$  corresponding to the  $(x+3)^2$  factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the  $\frac{1}{x^2}$  toolkit.

At the vertical asymptote  $x = 2$  corresponding to the  $(x - 2)$  factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the  $\frac{1}{x}$  toolkit.

### Example 9

Sketch a graph of  $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$ .

We can start our sketch by finding intercepts and asymptotes. Evaluating the function at zero gives the vertical intercept:

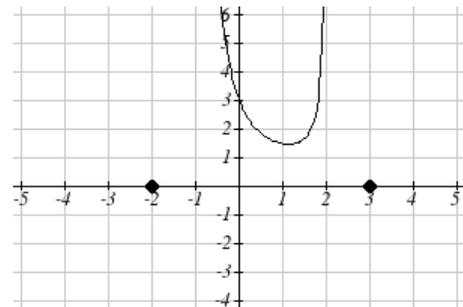
$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} = 3$$

Looking at when the numerator of the function is zero, we can determine the graph will have horizontal intercepts at  $x = -2$  and  $x = 3$ . At each, the behavior will be linear, with the graph passing through the intercept.

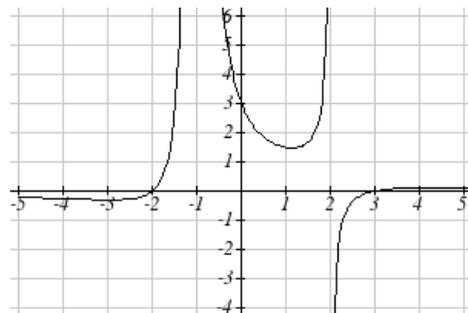
Looking at when the denominator of the function is zero, we can determine the graph will have vertical asymptotes at  $x = -1$  and  $x = 2$ .

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at  $y = 0$ .

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no horizontal intercepts between the vertical asymptotes, and the vertical intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph.



Since the factor associated with the vertical asymptote at  $x = -1$  was squared, we know the graph will have the same behavior on both sides of the asymptote. Since the graph heads towards positive infinity as the inputs approach the asymptote on the right, the graph will head towards positive infinity on the left as well. For the vertical asymptote at  $x = 2$ , the factor was not squared, so the graph will have opposite behavior on either side of the asymptote.



After passing through the horizontal intercepts, the graph will then level off towards an output of zero, as indicated by the horizontal asymptote.

## Try it Now

5. Given the function  $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$ , use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Since a rational function written in factored form will have a horizontal intercept where each factor of the numerator is equal to zero, we can form a numerator that will pass through a set of horizontal intercepts by introducing a corresponding set of factors. Likewise, since the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

## Writing Rational Functions from Intercepts and Asymptotes

If a rational function has horizontal intercepts at  $x = x_1, x_2, \dots, x_n$ , and vertical asymptotes at  $x = v_1, v_2, \dots, v_m$  then the function can be written in the form

$$f(x) = a \frac{(x - x_1)^{p_1} (x - x_2)^{p_2} \cdots (x - x_n)^{p_n}}{(x - v_1)^{q_1} (x - v_2)^{q_2} \cdots (x - v_m)^{q_m}}$$

where the powers  $p_i$  or  $q_i$  on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor  $a$  can be determined given a value of the function other than the horizontal intercept, or by the horizontal asymptote if it is nonzero.

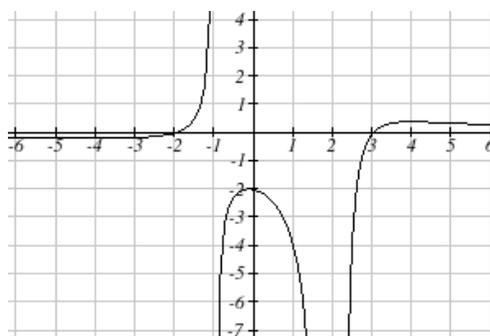
## Example 10

Write an equation for the rational function graphed here.

The graph appears to have horizontal intercepts at  $x = -2$  and  $x = 3$ . At both, the graph passes through the intercept, suggesting linear factors.

The graph has two vertical asymptotes. The one at  $x = -1$  seems to exhibit the basic

behavior similar to  $\frac{1}{x}$ , with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at  $x = 2$  is exhibiting a behavior similar to  $\frac{1}{x^2}$ , with the graph heading toward negative infinity on both sides of the asymptote.



Utilizing this information indicates an function of the form

$$f(x) = a \frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the vertical intercept (0,-2):

$$-2 = a \frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$

$$-2 = a \frac{-6}{4}$$

$$a = \frac{-8}{-6} = \frac{4}{3}$$

This gives us a final function of  $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$

### Important Topics of this Section

Inversely proportional; Reciprocal toolkit function

Inversely proportional to the square; Reciprocal squared toolkit function

Horizontal Asymptotes

Vertical Asymptotes

Rational Functions

Finding intercepts, asymptotes, and holes.

Given equation sketch the graph

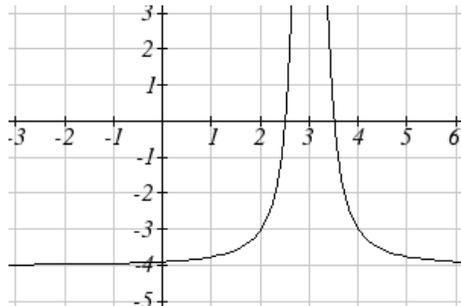
Identifying a function from its graph

### Try it Now Answers

1. Long run behavior, as  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow 0$

Short run behavior, as  $x \rightarrow 0$ ,  $f(x) \rightarrow \infty$  (there are no horizontal or vertical intercepts)

2.



The function and the asymptotes are shifted 3 units right and 4 units down.

As  $x \rightarrow 3$ ,  $f(x) \rightarrow \infty$  and as  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow -4$

3. Vertical asymptotes at  $x = 2$  and  $x = -3$ ; horizontal asymptote at  $y = 4$

4. For the transformed reciprocal squared function, we find the rational form.

$$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1 - 4(x-3)^2}{(x-3)^2} = \frac{1 - 4(x^2 - 6x + 9)}{(x-3)(x-3)} = \frac{-4x^2 + 24x - 35}{x^2 - 6x + 9}$$

Since the numerator is the same degree as the denominator we know that as  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow -4$ .  $y = -4$  is the horizontal asymptote. Next, we set the denominator equal to zero to find the vertical asymptote at  $x = 3$ , because as  $x \rightarrow 3$ ,  $f(x) \rightarrow \infty$ . We set the numerator equal to 0 and find the horizontal intercepts are at

$(2.5, 0)$  and  $(3.5, 0)$ , then we evaluate at 0 and the vertical intercept is at  $\left(0, \frac{-35}{9}\right)$

5.

Horizontal asymptote at  $y = 1/2$ .

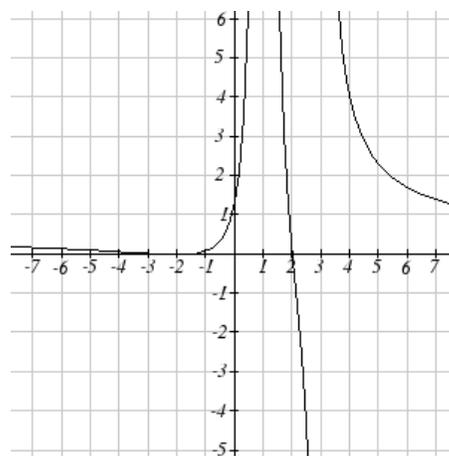
Vertical asymptotes are at  $x = 1$ , and  $x = 3$ .

Vertical intercept at  $(0, 4/3)$ ,

Horizontal intercepts  $(2, 0)$  and  $(-2, 0)$

$(-2, 0)$  is a double zero and the graph bounces off the axis at this point.

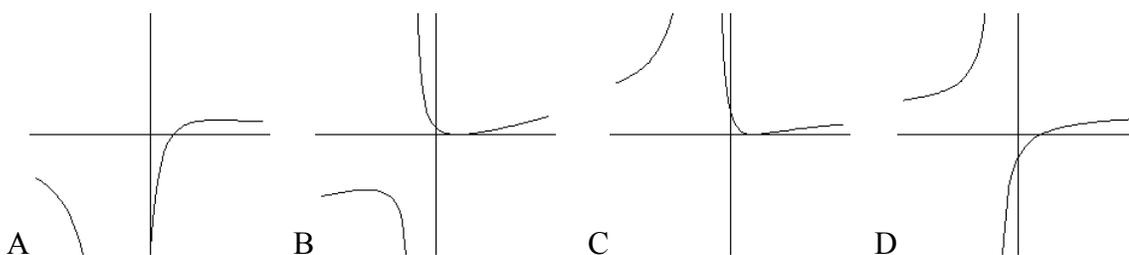
$(2, 0)$  is a single zero and crosses the axis at this point.



### Section 3.4 Exercises

Match each equation form with one of the graphs.

$$1. f(x) = \frac{x-A}{x-B} \quad 2. g(x) = \frac{(x-A)^2}{x-B} \quad 3. h(x) = \frac{x-A}{(x-B)^2} \quad 4. k(x) = \frac{(x-A)^2}{(x-B)^2}$$



For each function, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal asymptote. Use that information to sketch a graph.

$$5. p(x) = \frac{2x-3}{x+4}$$

$$6. q(x) = \frac{x-5}{3x-1}$$

$$7. s(x) = \frac{4}{(x-2)^2}$$

$$8. r(x) = \frac{5}{(x+1)^2}$$

$$9. f(x) = \frac{3x^2 - 14x - 5}{3x^2 + 8x - 16}$$

$$10. g(x) = \frac{2x^2 + 7x - 15}{3x^2 - 14x + 15}$$

$$11. a(x) = \frac{x^2 + 2x - 3}{x^2 - 1}$$

$$12. b(x) = \frac{x^2 - x - 6}{x^2 - 4}$$

$$13. h(x) = \frac{2x^2 + x - 1}{x - 4}$$

$$14. k(x) = \frac{2x^2 - 3x - 20}{x - 5}$$

$$15. n(x) = \frac{3x^2 + 4x - 4}{x^3 - 4x^2}$$

$$16. m(x) = \frac{5-x}{2x^2 + 7x + 3}$$

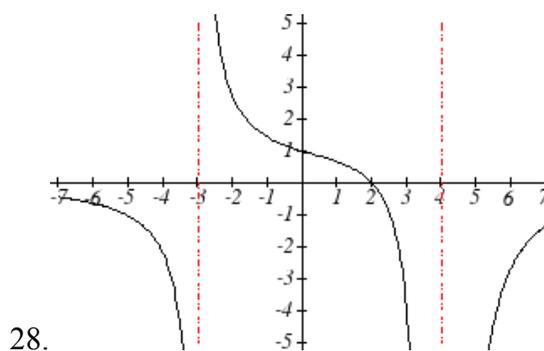
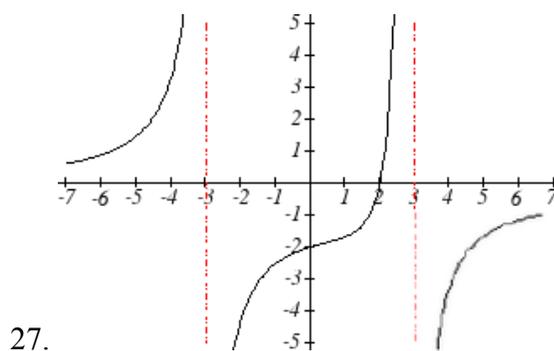
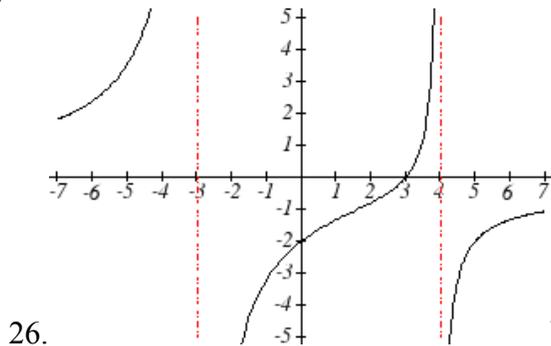
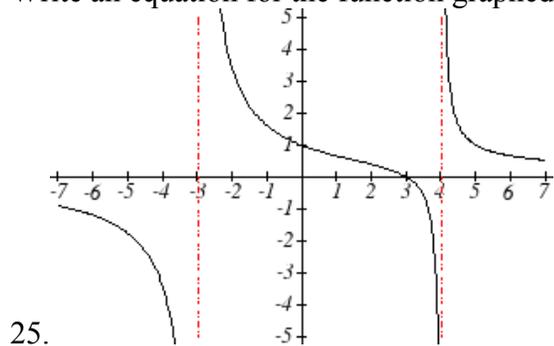
$$17. w(x) = \frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$$

$$18. z(x) = \frac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$$

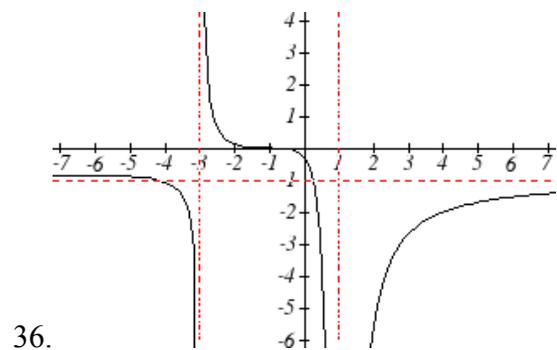
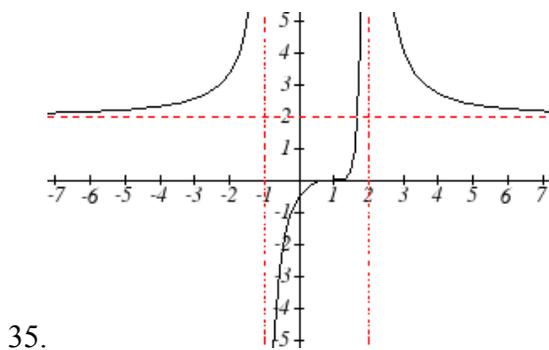
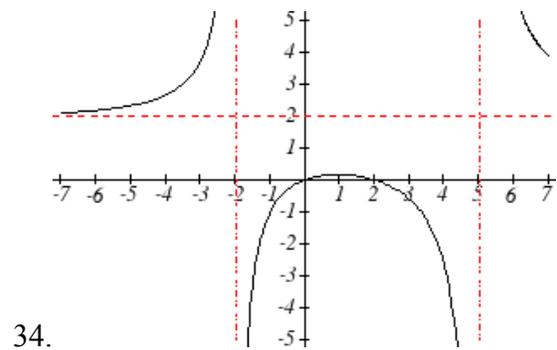
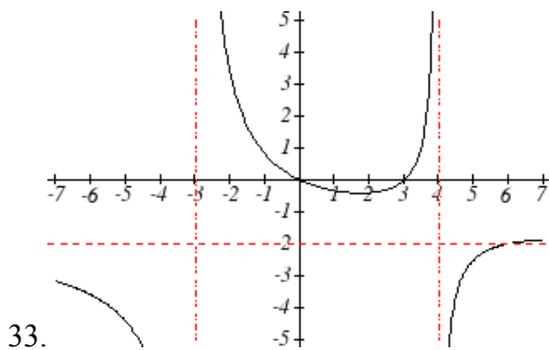
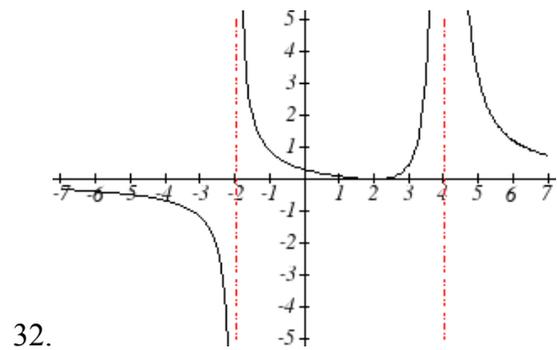
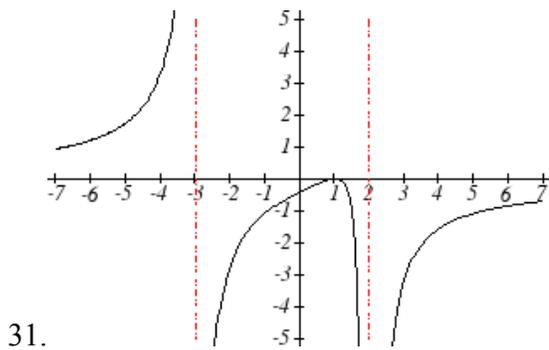
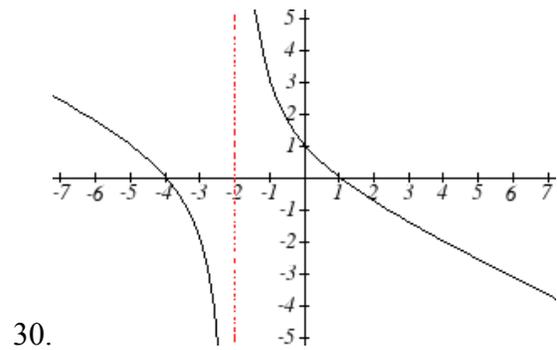
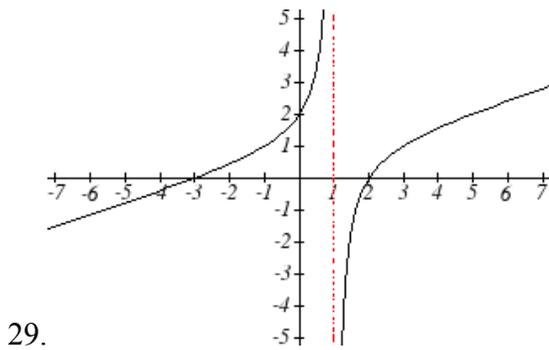
Write an equation for a rational function with the given characteristics.

19. Vertical asymptotes at  $x = 5$  and  $x = -5$   
 $x$  intercepts at  $(2,0)$  and  $(-1,0)$        $y$  intercept at  $(0,4)$
20. Vertical asymptotes at  $x = -4$  and  $x = -1$   
 $x$  intercepts at  $(1,0)$  and  $(5,0)$        $y$  intercept at  $(0,7)$
21. Vertical asymptotes at  $x = -4$  and  $x = -5$   
 $x$  intercepts at  $(4,0)$  and  $(-6,0)$       Horizontal asymptote at  $y = 7$
22. Vertical asymptotes at  $x = -3$  and  $x = 6$   
 $x$  intercepts at  $(-2,0)$  and  $(1,0)$       Horizontal asymptote at  $y = -2$
23. Vertical asymptote at  $x = -1$   
 Double zero at  $x = 2$        $y$  intercept at  $(0,2)$
24. Vertical asymptote at  $x = 3$   
 Double zero at  $x = 1$        $y$  intercept at  $(0,4)$

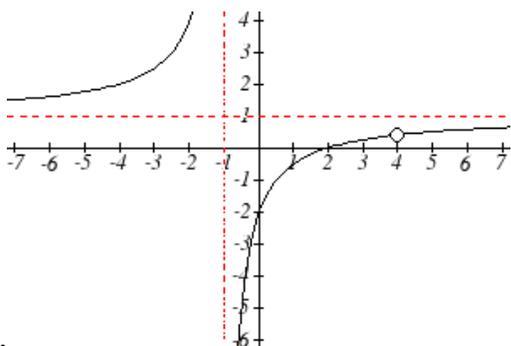
Write an equation for the function graphed.



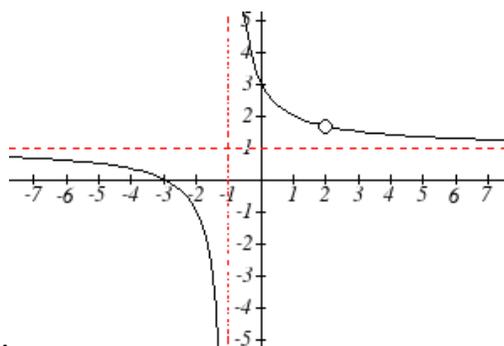
Write an equation for the function graphed.



Write an equation for the function graphed.



37.

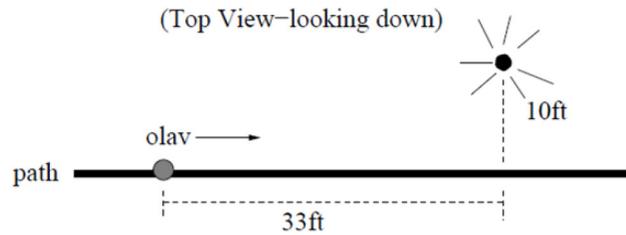


38.

39. A scientist has a beaker containing 20 mL of a solution containing 20% acid. To dilute this, she adds pure water.
- Write an equation for the concentration in the beaker after adding  $n$  mL of water.
  - Find the concentration if 10 mL of water has been added.
  - How many mL of water must be added to obtain a 4% solution?
  - What is the behavior as  $n \rightarrow \infty$ , and what is the physical significance of this?
40. A scientist has a beaker containing 30 mL of a solution containing 3 grams of potassium hydroxide. To this, she mixes a solution containing 8 milligrams per mL of potassium hydroxide.
- Write an equation for the concentration in the tank after adding  $n$  mL of the second solution.
  - Find the concentration if 10 mL of the second solution has been added.
  - How many mL of water must be added to obtain a 50 mg/mL solution?
  - What is the behavior as  $n \rightarrow \infty$ , and what is the physical significance of this?
41. Oscar is hunting magnetic fields with his gauss meter, a device for measuring the strength and polarity of magnetic fields. The reading on the meter will increase as Oscar gets closer to a magnet. Oscar is in a long hallway at the end of which is a room containing an extremely strong magnet. When he is far down the hallway from the room, the meter reads a level of 0.2. He then walks down the hallway and enters the room. When he has gone 6 feet into the room, the meter reads 2.3. Eight feet into the room, the meter reads 4.4. [UW]
- Give a rational model of form  $m(x) = \frac{ax+b}{cx+d}$  relating the meter reading  $m(x)$  to how many feet  $x$  Oscar has gone into the room.
  - How far must he go for the meter to reach 10? 100?
  - Considering your function from part (a) and the results of part (b), how far into the room do you think the magnet is?

42. The more you study for a certain exam, the better your performance on it. If you study for 10 hours, your score will be 65%. If you study for 20 hours, your score will be 95%. You can get as close as you want to a perfect score just by studying long enough. Assume your percentage score,  $p(n)$ , is a function of the number of hours,  $n$ , that you study in the form  $p(n) = \frac{an+b}{cn+d}$ . If you want a score of 80%, how long do you need to study? [UW]

43. A street light is 10 feet north of a straight bike path that runs east-west. Olav is bicycling down the path at a rate of 15 miles per hour. At noon, Olav is 33 feet west of the point on the bike path closest to the street light. (See the



picture). The relationship between the intensity  $C$  of light (in candlepower) and the distance  $d$  (in feet) from the light source is given by  $C = \frac{k}{d^2}$ , where  $k$  is a constant

depending on the light source. [UW]

- From 20 feet away, the street light has an intensity of 1 candle. What is  $k$ ?
- Find a function which gives the intensity of the light shining on Olav as a function of time, in seconds.
- When will the light on Olav have maximum intensity?
- When will the intensity of the light be 2 candles?